# Globally optimal SISO $H_2$-norm model reduction

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## 1 Introduction

State-of-the-art methods for $H_2$-norm model reduction have demonstrated exceptional performance on large-scale problems [4], however, the methods are generally suboptimal as only local optimality can be guaranteed. The approach that we present is, by contrast, deterministic in the sense that it ensures to identify the globally optimal reduced model. We use the first-order necessary conditions for optimality of the single-input single-output (SISO) $H_2$-norm model reduction problem, which can be shown to be interpolatory conditions (Walsh's theorem), to reformulate the problem as a multivariate root-finding problem. While aiming for global optimality through this reformulation comes at the cost of a combinatorial growth of the problem complexity for increasing model orders, our methodology allows to tackle larger problems compared to the other globally optimal approaches described in the literature [1, 2].

## 2 $H_2$-norm model reduction

Given a 'higher-order' system $H(z) \in H_2$, with $H_2$ the class of stable and causal LTI models with real-valued impulse response:

$$H(z) = \frac{b(z)}{a(z)} = \frac{b_{n-1}z^{n-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0},$$

model reduction techniques search for a model $\hat{H}(z) \in H_2$ of order $m < n$, while minimizing the approximation error in some measure. We consider the $H_2$-norm:

$$\begin{aligned}\min_{\hat{H}(z)} \quad & J = \left\|H(z) - \hat{H}(z)\right\|_{H_2}^2 \\ & = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega}) - \hat{H}(e^{j\omega})|^2 d\omega = \sum_{k=0}^{\infty} (h_k - \hat{h}_k)^2,\end{aligned}$$

with $\{h_k\}_{k=0,\ldots,\infty}$, $\{\hat{h}_k\}_{k=0,\ldots,\infty}$ the impulse response of $H(z)$ and $\hat{H}(z)$ respectively.

## 3 Walsh's theorem

Walsh's theorem originates from the field of rational approximation theory [9], but has also been derived in the context of systems and control [3, 6, 7]. The result states that in the stationary points of the model reduction problem $\hat{H}(z)$ satisfies:

$$H(z) - \hat{H}(z) = \frac{b(z)}{a(z)} - \frac{\hat{b}(z)}{\hat{a}(z)} = \left[\frac{\hat{a}_r(z)}{\hat{a}(z)}\right]^2 G(z),$$

with $G(z) \in H_2$ the $z$-transform of some real-valued, stable and causal signal and $\hat{a}_r(z)$ defined as $\hat{a}_r(z) = \hat{a}_0 z^m + \hat{a}_1 z^{m-1} + \cdots + \hat{a}_{m-1}z + 1$, the polynomial which has the reciprocals of the roots of $\hat{a}(z)$ as its roots.

## 4 Methodology

Manipulation of Walsh's relation allows to construct a square system of $n+m$ multivariate polynomial equations in the $2m$ decision variables $\{\hat{a}_i, \hat{b}_i\}_{i=0,\ldots,m-1}$ and $n-m$ auxiliary variables. After pruning the common-roots of the system (the $\{\hat{a}_i, \hat{b}_i\}_{i=0,\ldots,m-1}$ must be real-valued and $\hat{H}(z)$ stable), the globally optimal minimizer(s) of the model reduction problem can be selected by means of evaluation of $J$. The system contains one linear, one quadratic and $n+m-2$ cubic equations, the low degree of which is a major advantage as it keeps the complexity of the root-finding problem relatively low: other globally optimal approaches [1, 2] have been applied to setups with at most $(n,m) = (4,2)$, whereas e.g., the case $(15,4)$ seems feasible using our methodology.

Walsh's theorem has been considered in the context of the $H_2$-norm model reduction problem [6, 7], but, up to our knowledge, it has never been used to solve directly for the stationary points. We will present the theoretical machinery behind the novel methodology using a didactic toy problem and show how our approach, in combination with state-of-the-art root-finding methods [5, 8], can tackle numerical examples that were considered unfeasible before.

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